

Boson-Sampling with non-interacting fermions

V. S. Shchesnovich

Centro de Ciências Naturais e Humanas,

Universidade Federal do ABC, Santo André, SP, 09210-170 Brazil

Abstract

We explore the conditions under which identical particles in unitary linear networks behave as the other species, i.e. bosons as fermions and fermions as bosons. It is found that the Boson-Sampling computer of Aaronson & Arkhipov can be implemented in an interference experiment with non-interacting fermions in an appropriately entangled state. Moreover, a scheme is proposed which simulates the scattershot version of the Boson-Sampling computer by preparing, on the fly, the required entangled state of fermions from an unentangled one.

I. INTRODUCTION

The Boson-Sampling (BS) computer of S. Aaronson and A. Arkhipov [1] attracts attention of physicists due to two basic pillars: (i) simplicity, since only passive linear optical devices, bucket detectors, and single-photon sources are required for its experimental implementation, moreover, clearly not sufficient for the universal quantum computation (UQC) [2–4] and (ii) at the same time, due to interference of bosonic path amplitudes the output probability distribution of the BS computer lies in a higher-order $\#P$ class of computational complexity [5–7], asymptotically inaccessible for a classical computer even to verify the result. In more precise terms, Ref. [1] presents evidence that *even an approximate* classical simulation of the probability distribution in the BS computer output results in very drastic and utterly improbable consequences for the computational complexity theory. The optical implementation of the BS computer is based on completely indistinguishable single photons, producing the Hong-Ou-Mandel dip [8] (see also Ref. [9]), at a unitary network input and non-adaptive photon counting measurements at the network output. In practical terms, with few dozens of single photons the BS device would be more powerful a computer than our current computers [1]. Several labs have performed the proof of principle experiments on small networks with few single photons [10–14]. Scalability of a realistic BS computer is still under discussion [15, 16] due to unavoidable errors in any experimental realization. Several realistic error models have been analyzed [17–20]. Other setups were proposed [21–23] which could turn out to be of advantage in experimental realization. Though no practical problem is known to be solvable on the BS computer and its verification poses profound questions [24, 25], with the experimental efforts in this direction [26, 27] and search for conditional certification protocols [28] (see also Ref. [29]), such a device would undoubtedly have a profound impact on physics and technology.

In all discussions of the BS computer it is universally assumed that identical bosons are indispensable. It is usually argued that there are profound differences between complexity of behavior of non-interacting identical bosons and fermions in unitary networks: output probabilities of completely indistinguishable bosons are given by the absolute values squared of the matrix permanents, classically hard to compute [5, 7], whereas output probabilities of completely indistinguishable fermions are given by the absolute values squared of the matrix determinants, easily computable. Furthermore, for fermions there are no-go theorems [30–

34] stating that non-interacting identical fermions with only single-mode measurements, the so-called fermionic linear optics (FLO), do not represent any problem in terms of complexity for the classical computer. On the other hand, it was also shown that adding a two-mode electric charge measurement to the FLO allows for the UQC [35] (see also Ref. [34]).

The BS computer raises profound questions. What is the physical property of non-interacting indistinguishable bosons responsible for its computational complexity? Why non-interacting indistinguishable fermions do not display a similar complexity? Can one implement BS computer with non-identical quantum particles instead of identical ones? *Below we show that identical fermions and even non-identical quantum particles can be employed to emulate the BS computer distribution via particle counting measurement at a unitary network output when an appropriately entangled input state is used.* Moreover, we indicate a scheme emulating with fermions the scattershot BS computer of Ref. [22], which requires a non-entangled input state and is based on a non-demolition (on-off type) particle counting at an intermediate stage.

II. THE BS COMPUTER WITH IDENTICAL BOSONS

The BS computer proposed in Ref. [1] relies on a quantum multi-particle interference in an unitary linear network with single photons at input. For below, we will generalize slightly the setup and consider an arbitrary N -particle input with a certain number of bosons per input mode, say n_k for input k , in a M -mode unitary network with the transformation matrix U relating input a_k^\dagger and output b_l^\dagger boson creation operators

$$a_k^\dagger = \sum_{l=1}^M U_{kl} b_l^\dagger. \quad (1)$$

Let us denote a state from a Fock basis with occupation numbers $\vec{n} = (n_1, \dots, n_M)$ by $\|\vec{n}\rangle$, where a subscript a/b will indicate the input/output modes. The following Fock-space expansion of an input state $\|\vec{n}\rangle_a$ in terms of the output Fock states $\|\vec{m}\rangle_b$ is valid [1]

$$\|\vec{n}\rangle_a = \sum_{\vec{m}} \frac{\text{per}(U[\vec{n}|\vec{m}])}{\sqrt{\mu(\vec{n})\mu(\vec{m})}} \|\vec{m}\rangle_b, \quad |\vec{m}| = |\vec{n}| = N, \quad (2)$$

where $\mu(\vec{n}) = \prod_{i=1}^M n_i!$, $|\vec{n}| = n_1 + \dots + n_M$, $\text{per}(\dots)$ stands for the matrix permanent [36], and we denote by $U[\vec{n}|\vec{m}]$ the $N \times N$ -dimensional matrix obtained from U by taking the k th

row n_k times and the l th column m_l times (the order of rows/columns being unimportant). Expansion (2) considered for all inputs with $|\vec{n}| = N$ is an unitary transformation of the Fock states corresponding to the unitary transformation between the respective single-particle states, an equivalent of Eq. (1),

$$|k\rangle_a = \sum_{l=1}^M U_{kl} |l\rangle_b, \quad k = 1, \dots, M. \quad (3)$$

By Eq. (2) probability to observe an output configuration \vec{m} is

$$p_B(\vec{m}|\vec{n}) = |{}_b\langle\vec{m}|\vec{n}\rangle_a|^2 = \frac{|\text{per}(U[\vec{n}|\vec{m}])|^2}{\mu(\vec{n})\mu(\vec{m})}. \quad (4)$$

For single photons at input ($n_k \leq 1$) in the limit $M \gg N^2$ (the boson birthday paradox limit) [1, 37] we have $m_k \leq 1$ with the probability $1 - O(N^2/M)$. In this case the probability distribution at output of a network is given by the absolute values squared of the matrix permanents of submatrices of U which belong to the #P class of computational complexity [1]. It was noted recently that for $M \geq N^2$ the quantum many-body correlations of indistinguishable particles do not vanish in the thermodynamic limit [38].

The above account of the BS computer setup is, however, too idealistic: in practice, identical photons, besides the input modes (spatial [1] or time-bin [23]) that network transforms to output modes, have also other degrees of freedom, represented by their spectral states (see Ref. [29]). Correspondingly, a particle state is a tensor product of two factors: (i) a mode state $|k\rangle$ transformed by a network and (ii) internal state in the degrees of freedom not affected by the network, say $|\phi_\alpha\rangle$, $\alpha = 1, \dots, N$. The mode transformation effected by such a network must read

$$a_{k,\phi_\alpha}^\dagger = \sum_{l=1}^M U_{kl} b_{l,\phi_\alpha}^\dagger, \quad (5)$$

for $\alpha = 1, \dots, N$. For output probability to be given by Eq. (4) bosons (photons) should be completely indistinguishable (for more details, consult Ref. [29]). Below we explore the dependence of the probability distribution at output of an unitary linear network on the input state.

III. BOSONS BEHAVING AS FERMIONS AND FERMIONS AS BOSONS

We will deal with both bosons and fermions and even with non-identical particles, thus it is convenient to first define a combined set of notations which could be applied in all cases.

The first quantization representation for bosons and fermions is preferable in this respect, since it can describe also the states of non-identical particles.

A. First quantization representation and ε -symmetrization projectors

We divide particle observables into two classes: (i) the Hilbert space H of operating modes, acted on by a unitary network according to Eq. (3), and (ii) the Hilbert space \mathcal{H} of internal degrees of freedom not affected by the network. The single-particle Hilbert space is then a tensor product $H \otimes \mathcal{H}$, the single-particle states are denoted by $|k\rangle|\phi\rangle$, $|k\rangle \in H$ and $|\phi\rangle \in \mathcal{H}$ (the Roman letters will be used only for the mode states). Vector notations will be employed for N -particle states, i.e.

$$|\vec{k}\rangle \equiv \prod_{\alpha=1}^N |k_{\alpha}\rangle, \quad |\vec{\phi}\rangle \equiv \prod_{\alpha=1}^N |\phi_{\alpha}\rangle, \quad |\vec{k}, \vec{\phi}\rangle \equiv |\vec{k}\rangle|\vec{\phi}\rangle, \quad (6)$$

where $|\vec{k}\rangle \in H^{\otimes N}$ and $|\vec{\phi}\rangle \in \mathcal{H}^{\otimes N}$.

The states of N identical particles belong to the ε -symmetric subspace of the tensor product $H^{\otimes N} \otimes \mathcal{H}^{\otimes N}$, denoted by $\mathcal{S}_{\varepsilon}\{H^{\otimes N} \otimes \mathcal{H}^{\otimes N}\}$ below. Here $\mathcal{S}_{\varepsilon}$ is the projector on the ε -symmetric wave-function, where for bosons $\varepsilon(\sigma) = 1$ and for fermions $\varepsilon(\sigma) = \text{sgn}(\sigma)$ for a permutation σ , e.g. in $H^{\otimes N}$ it is defined as [39]

$$\mathcal{S}_{\varepsilon} \equiv \frac{1}{N!} \sum_{\sigma} \varepsilon(\sigma) P_{\sigma}, \quad P_{\sigma} |\vec{k}\rangle \equiv \prod_{\alpha=1}^N |k_{\sigma^{-1}(\alpha)}\rangle, \quad (7)$$

where the summation is over all permutations σ of N elements. The ε -symmetric state vectors will be denoted with a superscript “ (ε) ”, e.g., $|\vec{k}^{(\varepsilon)}\rangle \equiv \mathcal{S}_{\varepsilon} |\vec{k}\rangle$ and $|\vec{k}, \vec{\phi}^{(\varepsilon)}\rangle \equiv \mathcal{S}_{\varepsilon} |\vec{k}, \vec{\phi}\rangle$. Below we will use three types of the ε -symmetrization projectors: (i) acting on the whole space $H^{\otimes N} \otimes \mathcal{H}^{\otimes N}$ of N particles, (ii) on the mode space $H^{\otimes N}$ alone, or (iii) on the internal space $\mathcal{H}^{\otimes N}$ alone. These will be denoted by $\mathcal{S}_{\varepsilon}$, $\mathcal{S}_{\varepsilon} \otimes I$, and $I \otimes \mathcal{S}_{\varepsilon}$, respectively. When considering only the projector on the symmetric state we will replace “ ε ” by “ S ” (respectfully, by “ A ” for the projector on the anti-symmetric state). The same rule will apply for the ε -symmetric states.

We will denote the occupation number (Fock) states as follows

$$|\vec{n}^{(\varepsilon)}\rangle \equiv \sqrt{\frac{N!}{\mu(\vec{n})}} |\vec{k}^{(\varepsilon)}\rangle = \frac{1}{\sqrt{\mu(\vec{n})}} \prod_{\alpha=1}^N a_{k_{\alpha}}^{\dagger} |0\rangle, \quad (8)$$

where n_1, \dots, n_M are occupation numbers corresponding to the set of modes k_1, \dots, k_N (the same notations will be used for the bosonic and fermionic creation and annihilation operators). The state labels, e.g. k_1, \dots, k_N , have a fixed order in a tensor product of single-particle states to which \mathcal{S}_ε is applied, i.e. in $\mathcal{S}_\varepsilon|\vec{k}\rangle$, and in a product of creation operators, as in Eq. (8).

We consider N identical quantum particles in arbitrary internal states $|\phi_1\rangle, \dots, |\phi_N\rangle$, thus a state $\mathcal{S}_\varepsilon|\vec{k}, \vec{\phi}\rangle$ is, in general, a linear superposition of the Fock states in $\mathcal{S}_\varepsilon\{H^{\otimes N} \otimes \mathcal{H}^{\otimes N}\}$. However, the following relation, generalizing Eq. (8), is valid

$$\frac{1}{\sqrt{N!}} \prod_{\alpha=1}^N a_{k_\alpha, \phi_\alpha}^\dagger |0\rangle = \mathcal{S}_\varepsilon|\vec{k}, \vec{\phi}\rangle = |(\vec{k}, \vec{\phi})^{(\varepsilon)}\rangle. \quad (9)$$

(Eq. (9) can be verified by expansion of the internal state vector $|\phi_\alpha\rangle$ in some basis, using Eq. (8), and employing linearity of \mathcal{S}_ε and of the creation operators, i.e. $a_\chi^\dagger = c_1 a_\psi^\dagger + c_2 a_\phi^\dagger$ for $|\chi\rangle = c_1|\psi\rangle + c_2|\phi\rangle$).

The following important identity between the ε -symmetrization projectors in the Hilbert space $H^{\otimes N} \otimes \mathcal{H}^{\otimes N}$ will be heavily used

$$(I \otimes \mathcal{S}_{\varepsilon_2})\mathcal{S}_{\varepsilon_1} = (I \otimes \mathcal{S}_{\varepsilon_2})(\mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes I) = \mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes \mathcal{S}_{\varepsilon_2}. \quad (10)$$

Indeed, noticing that \mathcal{S}_ε acting in $H^{\otimes N} \otimes \mathcal{H}^{\otimes N}$ can be cast as $\mathcal{S}_\varepsilon = \frac{1}{N!} \sum_\sigma \varepsilon(\sigma) (P_\sigma \otimes I) (I \otimes P_\sigma)$ where $(P_\sigma \otimes I)$ and $(I \otimes P_\sigma)$ act in $H^{\otimes N}$ and $\mathcal{H}^{\otimes N}$, respectively, and that $P_{\sigma\tau} = P_\sigma P_\tau$ and $\varepsilon(\sigma\tau) = \varepsilon(\sigma)\varepsilon(\tau)$ for two permutations σ and τ we obtain

$$\begin{aligned} (I \otimes \mathcal{S}_{\varepsilon_2})\mathcal{S}_{\varepsilon_1} &= \left(\frac{1}{N!} \sum_\sigma \varepsilon_2(\sigma) (I \otimes P_\sigma) \right) \frac{1}{N!} \sum_{\sigma'} \varepsilon_1(\sigma') (P_{\sigma'} \otimes P_{\sigma'}) \\ &= \left(\frac{1}{N!} \right)^2 \sum_\sigma \sum_{\sigma'} \varepsilon_1(\sigma') \varepsilon_2(\sigma') \varepsilon_2(\sigma\sigma') (I \otimes P_{\sigma\sigma'}) (P_{\sigma'} \otimes I) \\ &= \left(\frac{1}{N!} \sum_{\sigma''} \varepsilon_2(\sigma'') (I \otimes P_{\sigma''}) \right) \frac{1}{N!} \sum_{\sigma'} \varepsilon_1(\sigma') \varepsilon_2(\sigma') (P_{\sigma'} \otimes I) \\ &= (I \otimes \mathcal{S}_{\varepsilon_2})(\mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes I), \end{aligned}$$

where we have used that $\varepsilon^2(\sigma) = 1$ and changed the summation from σ to $\sigma'' = \sigma\sigma'$. Note that identity (10) contrasts with the orthogonality property for the same space projectors, $\mathcal{S}_S \mathcal{S}_A = 0$. Identity (10) is the main reason allowing one to simulate bosonic behavior in a unitary linear network using entangled fermions and vice versa.

(i) Non-identical quantum particles:

$$\boxed{|1\rangle_a|\varphi\rangle \otimes |2\rangle_a|\phi\rangle}_{(1,1)} = \frac{1}{2} \left\{ \begin{array}{l} \boxed{|1\rangle_b|\varphi\rangle \otimes |1\rangle_b|\phi\rangle}_{(2,0)} \\ + \boxed{|2\rangle_b|\varphi\rangle \otimes |1\rangle_b|\phi\rangle - |1\rangle_b|\varphi\rangle \otimes |2\rangle_b|\phi\rangle}_{(1,1)} \\ - \boxed{|2\rangle_b|\varphi\rangle \otimes |2\rangle_b|\phi\rangle}_{(0,2)} \end{array} \right\}.$$

(ii) Identical particles (bosons for $\varepsilon_1 = 1$, fermions for $\varepsilon_1 = -1$):

$$\frac{1}{\sqrt{2}} \boxed{\begin{array}{l} |1\rangle_a|\varphi\rangle \otimes |2\rangle_a|\phi\rangle \\ + \varepsilon_1 |2\rangle_a|\phi\rangle \otimes |1\rangle_a|\varphi\rangle \end{array}}_{(1,1)} = \frac{1}{2\sqrt{2}} \left\{ \begin{array}{l} \boxed{|1\rangle_b|\varphi\rangle \otimes |1\rangle_b|\phi\rangle + \varepsilon_1 |1\rangle_b|\phi\rangle \otimes |1\rangle_b|\varphi\rangle}_{(2,0)} \\ + \boxed{\begin{array}{l} |2\rangle_b|\varphi\rangle \otimes |1\rangle_b|\phi\rangle - \varepsilon_1 |2\rangle_b|\phi\rangle \otimes |1\rangle_b|\varphi\rangle \\ - |1\rangle_b|\varphi\rangle \otimes |2\rangle_b|\phi\rangle + \varepsilon_1 |1\rangle_b|\phi\rangle \otimes |2\rangle_b|\varphi\rangle \end{array}}_{(1,1)} \\ - \boxed{|2\rangle_b|\varphi\rangle \otimes |2\rangle_b|\phi\rangle + \varepsilon_1 |2\rangle_b|\phi\rangle \otimes |2\rangle_b|\varphi\rangle}_{(0,2)} \end{array} \right\}.$$

(iii) Identical particles in an entangled state ($\varepsilon_2 = \pm 1$):

$$\frac{C}{\sqrt{2}} \boxed{\begin{array}{l} |1\rangle_a|\varphi\rangle \otimes |2\rangle_a|\phi\rangle \\ + \varepsilon_2 |1\rangle_a|\phi\rangle \otimes |2\rangle_a|\varphi\rangle \\ + \varepsilon_1 |2\rangle_a|\phi\rangle \otimes |1\rangle_a|\varphi\rangle \\ + \varepsilon_1 \varepsilon_2 |2\rangle_a|\varphi\rangle \otimes |1\rangle_a|\phi\rangle \end{array}}_{(1,1)} = \frac{C}{2\sqrt{2}} \left\{ \begin{array}{l} \boxed{(1 + \varepsilon_1 \varepsilon_2) |1\rangle_b|\varphi\rangle \otimes |1\rangle_b|\phi\rangle + (\varepsilon_1 + \varepsilon_2) |1\rangle_b|\phi\rangle \otimes |1\rangle_b|\varphi\rangle}_{(2,0)} \\ + \boxed{\begin{array}{l} (1 - \varepsilon_1 \varepsilon_2) |2\rangle_b|\varphi\rangle \otimes |1\rangle_b|\phi\rangle - (\varepsilon_1 - \varepsilon_2) |2\rangle_b|\phi\rangle \otimes |1\rangle_b|\varphi\rangle \\ - (1 - \varepsilon_1 \varepsilon_2) |1\rangle_b|\varphi\rangle \otimes |2\rangle_b|\phi\rangle + (\varepsilon_1 - \varepsilon_2) |1\rangle_b|\phi\rangle \otimes |2\rangle_b|\varphi\rangle \end{array}}_{(1,1)} \\ - \boxed{(1 + \varepsilon_1 \varepsilon_2) |2\rangle_b|\varphi\rangle \otimes |2\rangle_b|\phi\rangle + (\varepsilon_1 + \varepsilon_2) |2\rangle_b|\phi\rangle \otimes |2\rangle_b|\varphi\rangle}_{(0,2)} \end{array} \right\}.$$

FIG. 1: Diagram of output states for two particles in internal states $|\phi\rangle$ and $|\varphi\rangle$ sent through two different input modes ($|1\rangle_a$ and $|2\rangle_a$) of a balanced beam splitter. Here: (i) non-identical quantum particles, (ii) identical bosons ($\varepsilon_1 = 1$) or fermions ($\varepsilon_1 = -1$), and (iii) identical particles in a state symmetric ($\varepsilon_2 = 1$) or anti-symmetric ($\varepsilon_2 = -1$) with respect to the transposition of their internal states. Here $C = \frac{1}{\sqrt{2}}(1 + \varepsilon_2|\langle\varphi|\phi\rangle|^2)^{-1/2}$. Throughout, each box gives a certain particle configuration, indicated by a subscript, in input (a , on the left) or output (b , on the right) modes. The input state in each case is equal to a sum of all output states (from all output configurations).

B. Example of two quantum particles on a balanced beam splitter

Before considering the general case, let us analyze in some detail the simplest possible case of two particles at a balanced beam splitter:

$$|1\rangle_a = \frac{1}{\sqrt{2}} (|1\rangle_b + |2\rangle_b), \quad |2\rangle_a = \frac{1}{\sqrt{2}} (|1\rangle_b - |2\rangle_b). \quad (11)$$

We are interested in the effect of an input state ε -symmetric in the internal degrees of freedom on the output probability distribution. The results are presented in Figs. 1 and 2. Fig. 1

(a) Representation via the tensor product of the Hilbert spaces of modes and internal states:

$$\frac{C}{\sqrt{2}} \boxed{\begin{matrix} (|1\rangle_a|2\rangle_a + \varepsilon_1\varepsilon_2|2\rangle_a|1\rangle_a) \\ \otimes (|\varphi\rangle|\phi\rangle + \varepsilon_2|\phi\rangle|\varphi\rangle) \end{matrix}}_{(1,1)} = \frac{C}{\sqrt{2}} \left\{ \begin{aligned} & \boxed{\frac{\varepsilon_1\varepsilon_2+1}{2}|1\rangle_b|1\rangle_b \otimes (|\varphi\rangle|\phi\rangle + \varepsilon_2|\phi\rangle|\varphi\rangle)}_{(2,0)} \\ & + \boxed{\frac{\varepsilon_1\varepsilon_2-1}{2}(|1\rangle_b|2\rangle_b + \varepsilon_1\varepsilon_2|2\rangle_b|1\rangle_b) \otimes (|\varphi\rangle|\phi\rangle + \varepsilon_2|\phi\rangle|\varphi\rangle)}_{(1,1)} \\ & - \boxed{\frac{\varepsilon_1\varepsilon_2+1}{2}|2\rangle_b|2\rangle_b \otimes (|\varphi\rangle|\phi\rangle + \varepsilon_2|\phi\rangle|\varphi\rangle)}_{(0,2)} \end{aligned} \right\}.$$

(b) Equivalent representation using the Fock basis of mode states ($\varepsilon \equiv \varepsilon_1\varepsilon_2$):

$$\begin{aligned} \|(1,1)^{(\varepsilon)}\rangle_a \otimes |\Psi(\varphi, \phi)\rangle &= \left(\frac{\varepsilon+1}{2\sqrt{2}} \left[\|(2,0)^{(\varepsilon)}\rangle_b - \|(0,2)^{(\varepsilon)}\rangle_b \right] + \frac{\varepsilon-1}{2} \|(1,1)^{(\varepsilon)}\rangle_b \right) \otimes |\Psi(\varphi, \phi)\rangle, \\ |\Psi(\varphi, \phi)\rangle &\equiv C(|\varphi\rangle|\phi\rangle + \varepsilon_2|\phi\rangle|\varphi\rangle). \end{aligned}$$

FIG. 2: Case (iii) of Fig. 1 recast as a tensor product of the Hilbert spaces of modes and internal states, panel (a), and in terms of the occupation number (Fock) states in $\mathcal{S}_\varepsilon \{H^{\otimes N}\}$ with $\varepsilon = \varepsilon_1\varepsilon_2$, panel (b).

gives the output states in the first quantization representation, where we consider (i) non-identical particles in an unentangled state, with internal states $|\varphi\rangle$ and $|\phi\rangle$, (ii) unentangled identical particles in the same internal states, and (iii) identical particles in a generally entangled input state, ε -symmetric in the internal states. In Fig. 2 we first rewrite the state of case (iii) of Fig. 1 by separating in the tensor product the state of modes from the internal state, panel (a), and then cast the result in a more familiar Fock state representation using Eq. (8), panel (b). From the latter one can easily deduce the corresponding probabilities at the network output. From Fig. 2(b) it is seen that the effective bosonic behavior occurs for $\varepsilon = 1$ and the fermionic one for $\varepsilon = -1$, where $\varepsilon = \varepsilon_1\varepsilon_2$, the product of the “natural” symmetry of the input state with respect to the transposition of particles themselves, ε_1 , and the symmetry with respect to the transposition of the internal states, ε_2 (here ε_i is the

value $\varepsilon_i(\tau)$ where τ is the transposition of two objects). Note that the state given in Fig. 2 corresponds to the completely indistinguishable identical particles, since they behave as either indistinguishable identical bosons or fermions.

Thus, we can derive the following conclusion: an output probability of completely indistinguishable identical particles in an unitary network depends not on their natural symmetry ε_1 alone but on *the combined symmetry* $\varepsilon_1\varepsilon_2$ which includes as a factor the symmetry ε_2 with respect to permutations of their internal states. Below we show that this conclusion generalizes for N particles sent through an unitary network in an arbitrary input configuration.

C. States ε -symmetric in the internal degrees of freedom

Let us consider identical bosons first. For an output probability to be given by Eq. (4) bosons should be completely indistinguishable. One possibility is that their internal states are exactly the same $|\phi_\alpha\rangle = |\phi\rangle$, for $\alpha = 1, \dots, N$. But the latter is not the most general state of completely indistinguishable bosons, whereas the following permutation-symmetric input state is [29]

$$|\Psi_B^{(S)}(\vec{n})\rangle = \frac{c_S}{N!\sqrt{\mu(\vec{n})}} \sum_{\sigma} \prod_{\alpha=1}^N a_{k_\alpha, \phi_{\sigma(\alpha)}}^\dagger |0\rangle, \quad (12)$$

where the summation is over all permutations σ of N elements and $c_S^2 = N!/\text{per}(G)$ with $G_{\alpha\beta} \equiv \langle \phi_\alpha | \phi_\beta \rangle$.

Let us rewrite the state of Eq. (12) in the first quantization representation using the identities of Eqs. (8)–(10)

$$\begin{aligned} |\Psi_B^{(S)}(\vec{n})\rangle &= \frac{c_S \sqrt{N!}}{\sqrt{\mu(\vec{n})}} (I \otimes \mathcal{S}_S) \mathcal{S}_S |\vec{k}, \vec{\phi}\rangle_a = \frac{c_S \sqrt{N!}}{\sqrt{\mu(\vec{n})}} (\mathcal{S}_S \otimes \mathcal{S}_S) |\vec{k}\rangle_a |\vec{\phi}\rangle \\ &= c_S |\vec{n}^{(S)}\rangle_a |\vec{\phi}^{(S)}\rangle. \end{aligned} \quad (13)$$

Note that $\langle \vec{\phi}^{(S)} | \vec{\phi}^{(S)} \rangle = \langle \vec{\phi} | \mathcal{S}_S | \vec{\phi} \rangle = c_S^{-2}$. From the first quantization representation Eq. (13) is it clear why bosons in an input state of Eq. (12) must have output probabilities given by Eq. (4). Indeed, the unitary network acts in $H^{\otimes N}$, while leaving the internal space $\mathcal{H}^{\otimes N}$ invariant, therefore the output state is obtained by using the expansion in Eq. (2), resulting in the output distribution given by Eq. (4). The same conclusion is derived by simply noting the following mathematical identity between the symmetric states of modes, which follows

from Eq. (3)

$$|\vec{k}^{(S)}\rangle_a = \sum_{l_1 \leq \dots \leq l_N} \frac{\text{per}(U[\vec{n}|\vec{m}])}{\mu(\vec{m})} |\vec{l}^{(S)}\rangle_b. \quad (14)$$

Now turning to fermions, we note that, similar to Eq. (14), there is an identity relating two anti-symmetric states of modes

$$|\vec{k}^{(A)}\rangle_a = \sum_{l_1 < \dots < l_N} \det(U[\vec{n}|\vec{m}]) |\vec{l}^{(A)}\rangle_b, \quad (15)$$

where $k_1 < \dots < k_N$ and $l_1 < \dots < l_N$ and the corresponding occupation numbers satisfy $n_i, m_i \leq 1$ [49]. The identities (14) and (15) are equivalent forms of the corresponding expansions for Fock states (8) of modes, for bosons given by Eq. (2), whereas in case of fermions one must replace the matrix permanent by the matrix determinant.

Now let us analyze the general case of an input state similar to that of Eqs. (12)-(13). Consider an input state (bosons or fermions) of the following form

$$|\Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n})\rangle = c_{\varepsilon_2} \sqrt{\frac{N!}{\mu(\vec{n})}} (I \otimes \mathcal{S}_{\varepsilon_2}) \mathcal{S}_{\varepsilon_1} |\vec{k}, \vec{\phi}\rangle_a = c_{\varepsilon_2} \|\vec{n}^{(\varepsilon_1 \varepsilon_2)}\rangle_a |\vec{\phi}^{(\varepsilon_2)}\rangle, \quad (16)$$

where we have used Eqs. (10) and (8). The normalization factor is obviously given by

$$c_{\varepsilon_2}^{-2} = \langle \vec{\phi}^{(\varepsilon_2)} | \vec{\phi}^{(\varepsilon_2)} \rangle = \frac{1}{N!} \sum_{\sigma} \varepsilon_2(\sigma) \prod_{\alpha=1}^N \langle \phi_{\alpha} | \phi_{\sigma(\alpha)} \rangle, \quad (17)$$

i.e. proportional either to the matrix permanent for $\varepsilon_2(\sigma) = 1$, or to the matrix determinant for $\varepsilon_2(\sigma) = \text{sgn}(\sigma)$. The state of Eqs. (12)-(13) is just a special case of the state (16) for $\varepsilon_1 = \varepsilon_2 = 1$. Observe the following symmetry of the state in Eq. (16) due to the identity (10)

$$(P_{\sigma} \otimes I) |\Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n})\rangle = \varepsilon_1(\sigma) \varepsilon_2(\sigma) |\Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n})\rangle, \quad (\mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes I) |\Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n})\rangle = |\Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n})\rangle. \quad (18)$$

In section IIID we discuss in some detail the particle counting measurement and show that the input state (16) results in the bosonic behavior for $\varepsilon_1 \varepsilon_2 = 1$ and in the fermionic one for $\varepsilon_1 \varepsilon_2 = \text{sgn}$, the corresponding output distributions being given by the matrix permanents and by the matrix determinants, respectively. However, the input state (16) lies not in the $\varepsilon_1 \varepsilon_2$ -symmetric but in the ε_1 -symmetric subspace $\mathcal{S}_{\varepsilon_1} \{H^{\otimes N} \otimes \mathcal{H}^{\otimes N}\}$, i.e. it is a state of identical bosons for $\varepsilon_1 = 1$ and identical fermions for $\varepsilon_1 = \text{sgn}$. Indeed, by using Eq. (9) the state of Eq. (16) can be cast also as

$$|\Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n})\rangle = \frac{c_{\varepsilon_2}}{N! \sqrt{\mu(\vec{n})}} \sum_{\sigma} \varepsilon_2(\sigma) \prod_{\alpha=1}^N a_{k_{\alpha}, \phi_{\sigma(\alpha)}}^{\dagger} |0\rangle, \quad (19)$$

TABLE I: The ε -symmetric states and the interference behavior of identical particles.

Species \ Symmetry	Symmetric [$\varepsilon_2(\sigma) = 1$]	Anti-symmetric [$\varepsilon_2(\sigma) = \text{sgn}(\sigma)$]
Bosons [$\varepsilon_1(\sigma) = 1$]	$ \Psi_S^{(S)}(\vec{n})\rangle = c_S \ \vec{n}^{(S)}\rangle_a \vec{\phi}^{(S)}\rangle$ [Bosonic behavior]	$ \Psi_S^{(A)}(\vec{n})\rangle = c_A \ \vec{n}^{(A)}\rangle_a \vec{\phi}^{(A)}\rangle$ [Fermionic behavior]
Fermions [$\varepsilon_1(\sigma) = \text{sgn}(\sigma)$]	$ \Psi_A^{(S)}(\vec{n})\rangle = c_S \ \vec{n}^{(A)}\rangle_a \vec{\phi}^{(S)}\rangle$ [Fermionic behavior]	$ \Psi_A^{(A)}(\vec{n})\rangle = c_A \ \vec{n}^{(S)}\rangle_a \vec{\phi}^{(A)}\rangle$ [Bosonic behavior]

where the creation operators are bosonic for $\varepsilon_1 = 1$ and fermionic for $\varepsilon_1 = \text{sgn}$. From Eqs. (16) and (19) it is seen that for identical particles to behave as the other species, i.e. bosons as fermions and fermions as bosons, the input state must be anti-symmetric with respect to their internal states (the single-particle states $|\phi_1\rangle, \dots, |\phi_N\rangle$ must be linearly independent), which is an entangled state [45, 46]. In contrast, the particles show their “natural” behavior in an input state symmetric with respect to their internal states, which includes as a special case the unentangled state of all particles being in the same internal state. The four possible types of input states of Eqs. (16) or (19) of two particle species and two symmetries with respect to permutations of their internal states vs. the particle behavior are given in Table I.

Finally, we note that our method of emulating fermionic/bosonic behavior with bosons/fermions in a *single network* differs from that of Ref. [43] which requires N identical networks.

D. Particle counting measurement and output probabilities

Above it was assumed that the Fock representation $\|\vec{n}^{(\varepsilon_1 \varepsilon_2)}\rangle$ for the operational modes of an input state given by Eq. (16) results in the same probability distribution at a network output as that of the identical particles with the symmetry $\varepsilon_1 \varepsilon_2$. Since the symmetry of the Fock state does not always coincide with the “natural” symmetry of the identical particles this is not obvious. Let us analyze why this so in some detail. First, we assume that particle detectors are not distinguishing between various internal states (for instance, the spectral states in the case of photons). Second, we consider particle counting as ideal, i.e. without false counts or particle losses. Under these assumptions, let us derive the

corresponding positive operator-valued measure (POVM) describing such particle counting measurement, where one detects the number of particles in each output mode of a network without distinguishing between different internal states of the particles. For N identical particles, an element $\Pi^{(\varepsilon)}(\vec{m})$ of such a POVM corresponding to an output configuration \vec{m} reads (see also appendix A of Ref. [20] for the case of photons)

$$\Pi^{(\varepsilon)}(\vec{m}) = \frac{1}{\mu(\vec{m})} \sum_{\vec{j}} \left[\prod_{\alpha=1}^N b_{l_\alpha, j_\alpha}^\dagger \right] \|0\rangle \langle 0| \left[\prod_{\alpha=1}^N b_{l_\alpha, j_\alpha} \right] = \mathcal{S}_\varepsilon \Pi_{\vec{l}} \mathcal{S}_\varepsilon, \quad (20)$$

where $|j\rangle$, $j = 1, 2, 3, \dots$, is a basis of the internal single-particle space \mathcal{H} , \vec{l} is a vector of output modes corresponding to the configuration \vec{m} , and $\Pi_{\vec{l}}$ is defined as follows

$$\Pi_{\vec{l}} \equiv \frac{N!}{\mu(\vec{m})} |\vec{l}\rangle_b {}_b \langle \vec{l}| \otimes I. \quad (21)$$

Indeed, using identity (9) one can prove equivalence of the two representations on the r.h.s. of Eq. (20). The second form of the operator $\Pi^{(\varepsilon)}(\vec{m})$ in Eq. (20) follows from the fact that it is positive and non-distinguishing between the internal states, thus $\mathcal{S}_\varepsilon |\vec{l}, \vec{j}\rangle$ is its eigenstate for an eigenvalue which depends only on the corresponding output configuration \vec{m} . The factor (eigenvalue) $\frac{N!}{\mu(\vec{m})}$ is obtained by the following observation

$$\sum_{|\vec{m}|=N} \Pi_{\vec{l}} = \sum_{\vec{l}} \frac{\mu(\vec{m})}{N!} \Pi_{\vec{l}} = I \otimes I, \quad (22)$$

where the summation runs over all vectors \vec{l} with $1 \leq l_\alpha \leq M$ varying independently for each α . Then from Eqs. (20) and (22) we obtain

$$\sum_{|\vec{m}|=N} \Pi^{(\varepsilon)}(\vec{m}) = \mathcal{S}_\varepsilon, \quad (23)$$

i.e. the identity operator in the ε -symmetric subspace $\mathcal{S}_\varepsilon \{H^{\otimes N} \otimes \mathcal{H}^{\otimes N}\}$ corresponding to identical particles.

The output probability of a configuration \vec{m} for an arbitrary input $|\Psi_\varepsilon(\vec{n})\rangle \in \mathcal{S}_\varepsilon \{H^{\otimes N} \otimes \mathcal{H}^{\otimes N}\}$ reads

$$p_\varepsilon(\vec{m}|\vec{n}) = \langle \Psi_\varepsilon(\vec{n}) | \Pi^{(\varepsilon)}(\vec{m}) | \Psi_\varepsilon(\vec{n}) \rangle = \langle \Psi_\varepsilon(\vec{n}) | \Pi_{\vec{l}} | \Psi_\varepsilon(\vec{n}) \rangle, \quad (24)$$

where the second form takes into account that $|\Psi_\varepsilon(\vec{n})\rangle$ is a ε -symmetric state (thus the projector \mathcal{S}_ε in the second form of the detection operator in Eq. (20) is redundant). Now,

substituting the input state of Eq. (16) into Eq. (24) and taking into account the symmetry (18) we obtain

$$\begin{aligned} p_{\varepsilon_1}(\vec{m}|\vec{n}) &= \langle \Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n}) | \Pi_{\vec{l}} | \Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n}) \rangle = \langle \Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n}) | (\mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes I) \Pi_{\vec{l}} (\mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes I) | \Psi_{\varepsilon_1}^{(\varepsilon_2)}(\vec{n}) \rangle \\ &= |{}_b \langle \vec{m}^{(\varepsilon_1 \varepsilon_2)} | \vec{n}^{(\varepsilon_1 \varepsilon_2)} \rangle_a|^2, \end{aligned} \quad (25)$$

since by Eqs. (8) and (21)

$$(\mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes I) \Pi_{\vec{l}} (\mathcal{S}_{\varepsilon_1 \varepsilon_2} \otimes I) = \frac{N!}{\mu(\vec{n})} |\vec{l}^{(\varepsilon_1 \varepsilon_2)}\rangle_b {}_b \langle \vec{l}^{(\varepsilon_1 \varepsilon_2)}| \otimes I = \|\vec{m}^{(\varepsilon_1 \varepsilon_2)}\rangle_b {}_b \langle \vec{m}^{(\varepsilon_1 \varepsilon_2)}\| \otimes I. \quad (26)$$

One of conclusions to derive from the above analysis is this: The probability of an output configuration \vec{m} in case of identical particles in an input state of Eq. (16) is the same as of non-identical ones in the same input state in the first quantization representation. Indeed, if one counts non-identical quantum particles and then simply “forgets” the particle labels, thus summing up identical output probabilities from all POVM elements $\Pi_{\vec{l}}$ corresponding to the same output configuration \vec{m} , one gets the same probability as for identical particles. We see that for emulation of the behavior of completely indistinguishable identical particles one needs an entangled state of non-identical particles. However, using the state of Eq. (16) is not the simplest way to emulate indistinguishable identical particles with non-identical ones, one may as well employ a state of the form $\left\{ \sqrt{\frac{N!}{\mu(\vec{n})}} \mathcal{S}_{\varepsilon} |\vec{k}\rangle_a \right\} |\phi\rangle^{\otimes N}$, i.e. an equivalent of the Fock state (8). The main problem in such an emulation would be the process which supplies the required entangled state.

IV. THE SCATTERSHOT BS COMPUTER WITH FERMIONS

Implementation of the BS computer with fermions requires generation of an entangled input state of N particles $|\Psi_A^{(A)}(\vec{n})\rangle$ of Eq. (16) with the input configuration \vec{n} satisfying $n_{\alpha} \leq 1$, $\alpha = 1, \dots, N$. If one is not able to produce such an entangled state by means of the FLO and some particle counting measurements, the above boson-fermion duality would be just a curious feature with no pathway to implement the BS computer with identical fermions. Below we show that there is at least one possibility to engineer the required entangled state of fermions where the key role is played by the boson birthday paradox.

It is well-known that the FLO is very limited in its multi-particle entangling power. For a general N -particle (entangled) state there seems to be a no-go theorem due to a very

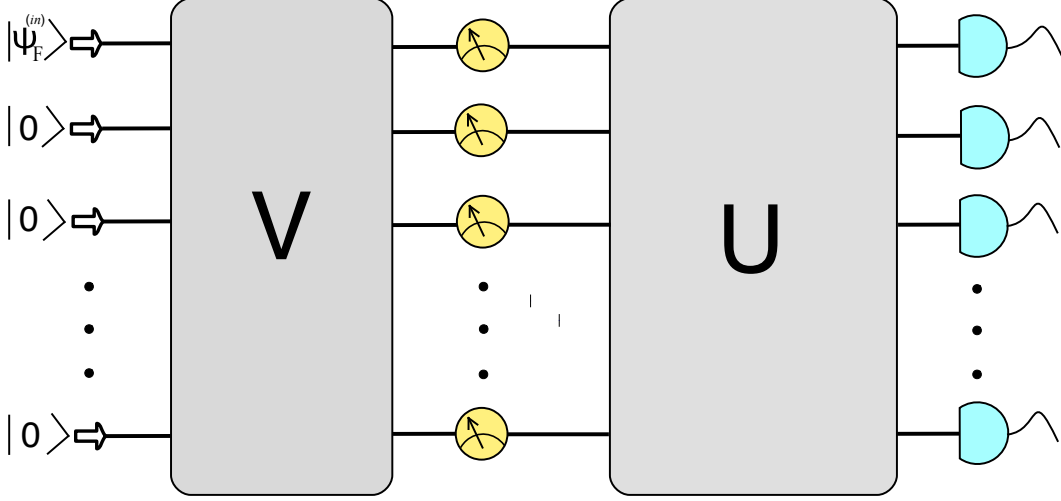


FIG. 3: Schematic depiction of the BS computer with fermions. The N -fermion Fock state Eq. (27) is at input mode $s = 1$ of a M -mode network V with $|V_{1,k}| = 1/\sqrt{M}$, whereas at other input modes of V is vacuum. Non-absorbing particle detectors, which do not distinguish between the internal states of fermions, are placed between the network V and a Haar-random network U . They register which of the input modes of U contain a fermion. Particle detectors at output modes of the network U sample the BS output distribution averaged over the input modes, i.e. the scheme implements the scattershot BS computer of Ref. [22].

unfavorable scaling of free parameters with N [44]. But, surprisingly, one can prepare the needed antisymmetric entangled state $|\Psi_A^{(A)}(\vec{n})\rangle$ from the following unentangled (see Refs. [45, 46]) N -fermion Fock state

$$|\Psi_F^{(in)}\rangle = [\det(G)]^{-\frac{1}{2}} \prod_{\alpha=1}^N d_{1,\phi_\alpha}^\dagger |0\rangle, \quad (27)$$

where the operator $d_{1,\phi_\alpha}^\dagger$ creates a fermion in some operational mode with index 1 and in an internal state $|\phi_\alpha\rangle$, and $G_{\alpha\beta} = \langle\phi_\alpha|\phi_\beta\rangle$. To this goal one needs to use an auxiliary M -mode unitary network, whose matrix V is assumed to satisfy the Fourier type condition for the first row $|V_{1k}| = 1/\sqrt{M}$. The state of Eq. (27) can be rewritten as follows $|\Psi_F^{(in)}\rangle = c_A \|\vec{q}^{(S)}\rangle_d |\vec{\phi}^{(A)}\rangle$, with $q_\alpha = N\delta_{\alpha,1}$ and c_A being the normalization of the internal state $|\vec{\phi}^{(A)}\rangle$ (given by Eq. (17)). According to Eqs. (2) or (14), the state (27) is transformed by the network V to a state whose expansion over the Fock states $\|\vec{n}^{(A)}\rangle_a$ (such that $|\vec{n}| = N$)

reads

$$|\Psi_F^{(in)}\rangle = c_A |\vec{q}^{(S)}\rangle_d |\vec{\phi}^{(A)}\rangle = c_A \sum_{\vec{n}} \sqrt{\frac{N!}{\mu(\vec{n})}} \left[\prod_{k=1}^M (V_{1,k})^{n_k} \right] |\vec{n}^{(A)}\rangle_a |\vec{\phi}^{(A)}\rangle. \quad (28)$$

Now assume that the network V satisfies the boson birthday paradox condition $M \gg N^2$. In this case, we have almost surely the Fock states on the r.h.s. of Eq. (28), at output of the network V , to satisfy $n_k \leq 1$. Indeed, from Eq. (28) we have probability for each particular output with $n_k \leq 1$ (i.e. $k_\alpha \neq k_\beta$ for $\alpha \neq \beta$)

$$P(k_1, \dots, k_N) = N! \prod_{\alpha=1}^N |V_{1,k_\alpha}|^2 = \frac{N!}{M^N}. \quad (29)$$

We get a non-bunched state over random N output modes of the network V with the probability $P_{BS} = \sum_{\vec{k}} P(k_1, \dots, k_N) = \frac{(M-N+1)!}{N!(M-N)!} \frac{N!}{M^N} = \prod_{q=1}^{N-1} (1 - \frac{q}{M}) \approx 1 - \frac{N(N-1)}{2M}$ [50]. The Fourier-type condition $|V_{1k}| = 1/\sqrt{M}$ corresponds to a local maximum of P_{BS} as function of V .

Since the modes containing a fermion at output of the network V are a random set, the scheme using such a network can simulate the scattershot BS computer of Ref. [22] with fermions in the input state (27), where the boson birthday paradox condition is required also for the BS computer itself [1]. To this goal one must detect a presence of a fermion in an output mode of the V -network without disturbing the internal N -particle state, unaffected by V . This is possible, at least in principle, by using non-absorbing particle detectors that do not distinguish between the internal states, as discussed in section IIID. Indeed, non-absorbing particle detection of fermions in output modes of V is described by the POVM of Eq. (20) where $\varepsilon = \text{sgn}$. The possibility of the non-disturbing detection is guaranteed by the following commutation rule

$$(I \otimes \mathcal{S}_A) \Pi^{(A)}(\vec{m}) = \Pi^{(A)}(\vec{m}) (I \otimes \mathcal{S}_A), \quad (30)$$

which can be easily established from the second form of $\Pi^{(A)}(\vec{m})$ on the r.h.s. of Eq. (20) [51].

Therefore, we have at least one scheme for implementation of the BS computer with identical fermions, depicted in Fig. 3. In one respect the scheme of Fig. 3 is very similar to the scattershot BS computer with photons proposed in Ref. [22], where an additional averaging over random, but known in each run, non-vacuum input modes of the network is implemented. Indeed, the BS computer with fermions of Fig. 3 has almost surely N non-vacuum input modes at network U , each containing a single fermion. The indices of the

non-vacuum input modes constitute a random set *known in each run*, where each particular configuration is generated with the same probability. The fact that the non-vacuum mode indices of the network U are known in each run leads to the same complexity of its output probability distribution as of the original BS computer with a fixed set of non-vacuum input modes [1, 22].

V. CONCLUSION

We have found the boson-fermion duality in unitary linear networks, i.e. bosons behaving as fermions and fermions as bosons in the appropriately entangled input states, allowing for the BS computer with non-interacting fermions. Such a possibility provides an insight on the physical origin of its computational complexity. Indeed, it was previously believed that only non-interacting identical bosons possess a fundamental feature allowing one to implement the BS computer. It is shown here that one can substitute bosons with non-interacting identical fermions in the antisymmetric entangled state over their internal degrees of freedom not affected by a network. Moreover, such an entangled N -particle state can be engineered by means of the FLO and a non-absorbing multi-mode particle counting measurement from a state of N fermions sharing a common (operational) mode and distributed over linearly-independent internal degrees of freedom. This agrees with the previous result [35] that the FLO can be promoted to a higher computational complexity, in our case to sampling from a #P-class problem, by using multi-mode non-absorbing particle counting.

Our goal was a proof of principle of the BS computer with fermions, an experimental implementation using currently available technology is challenging. However, there is an important progress in this direction – recent observation of the Hong-Ou-Mandel effect with massive identical particles, with fermions (electrons) [47] and with bosons (Helium atoms) [48]. We have not touched upon in our discussion which degrees of freedom of fermions could serve as their internal modes and, respectively, which would be the operational modes transformed by a network (and how to build such a network). Such questions and practical ways to implement the BS computer with fermions are left for the future research (in this respect, there are now already two different implementations for the BS computer with photons [1, 22] and one with excitation quasiparticles in trapped ions [21]). As a byproduct, we have also found how to implement the recently experimentally demonstrated method of

emulation of the Fermi-Dirac statistics with bosons [43] in a single network, instead of N identical ones for N particles, by entangling the internal states of particles instead of the operating modes transformed by a network.

Finally, from our discussion it follows that one can also simulate the behavior of identical particles with non-identical ones. For instance, the famous Hong-Ou-Mandel effect can be observed with non-identical particles in a single network, if the input state is a properly entangled one.

VI. ACKNOWLEDGEMENTS

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- [51] Here we note that an identity similar to Eq. (30) is valid for bosons. More generally, in both cases, $\Pi^{(\varepsilon_1)}(\vec{m})$ commutes also with $I \otimes \mathcal{S}_{\varepsilon_2}$ due to the second form of $\Pi^{(\varepsilon_1)}(\vec{m})$ on the r.h.s. of Eq. (20).